Development of Calculus in India: Contribution of Kerala School (1350–1550 CE)

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Way back in the 14th century, the mathematicians and astronomers residing on the banks of the river Nīlā in the south Malabar region of Kerala—in the context of finding the exact relationship between the circumference and the diameter of a circle, as also that between an arc and the corresponding chord of a circle—developed several ideas and techniques of what goes by the name of infinitesimal calculus today. In fact, they had advanced to the point of discovering the series expansions of the sine, cosine and arctangent functions. It has now been generally recognized that these achievements of the Kerala School, are in fact very much in continuation with the earlier work of Indian mathematicians, especially of the *Āryabhaṭan* school, during the period 500–1350 CE [4,5,6].

The Kerala School, pioneered by Mādhava (c. 1340–1420) and followed by illustrious mathematicians and astronomers like Parameśvara, Dāmodara, Nīlakaṇṭha, Acyuta and others, extended well into the 19th century as exemplified in the work of Śaṅkaravarman (c. 1830). Only a couple of astronomical works of Mādhava seem to be extant now. Most of his celebrated mathematical discoveries—such as the infinite series for π, its fast convergent approximations and so on—are available only in the form of citations in later works. Mādhava's disciple Parameśvara (c. 1380–1460) is reputed to have carried out detailed observations for over 50 years and composed a large number of original works and commentaries.

Nīlakaṇṭha (c. 1444–1550), disciple of Parameśvara's son Dāmodara (c. 1410–1520) and the author of *Tantrasaṅgraha* and *Āryabhaṭīya-bhāṣya*, is the most celebrated member of Kerala School after Mādhava. Apart from expounding in detail on the mathematical discoveries of Mādhava, Nīlakaṇṭha also came up with a remarkable revision of the traditional Indian planetary model which—for the first time in the history of astronomy—gives the correct formulation of the equation of centre and latitudinal motion of the interior planets [8].

A systematic exposition of the work of the Kerala School, is to be found in the famous Malayalam work *Gaṇita-yukti-bhāṣā* (Rationales in Mathematical Astronomy) composed by Jyeṣṭhadeva (c. 1530)—a disciple of Dāmodara and junior to Nīlakaṇṭha. Another detailed exposition is available in the Sanskrit commentaries written by Śaṅkara Vāriyar (c. 1500–1550): *Kriyākramakarī* on Līlāvatī of Bhāskara and *Yukti-dīpikā* on *Tantrasaṅgraha* of Nīlakaṇṭha. The scope of the present article is confined to provide a brief overview based on the exposition given in *Yuktibhāṣā* with occasional references to the other works such as *Āryabhaṭīya-bhāṣya* and *Kriyākramakarī*.

To provide a glimpse of some of the concepts and methods developed by the Kerala mathematicians, we start our discussions with the issue of irrationality of π and the summation of infinite geometric series as presented by Nīlakaṇṭha in his *Āryabhaṭīya-bhāṣya*. We then consider the derivation of binomial series expansion and the estimation of the sum of integral powers of integers, as discussed in *Yuktibhāṣā*. These results constitute the basis for the derivation of the infinite series for $\frac{\pi}{4}$ due to Mādhava. We shall outline this derivation as also the very interesting work of Mādhava on the estimation of the end-correction terms to find accurate approximations to π. In the final section, we shall deal with another topic which has a bearing on calculus, but is not dealt with in *Yuktibhāṣā*, namely the evaluation of the instantaneous velocity of a planet. Here, we shall present the formula for the instantaneous velocity of a planet given by Nīlakaṇṭha which involves the derivative of the arcsine function.

Irrationality of π

Having specified the ratio of the circumference to the diameter of a circle, Āryabhaṭa in his *Āryabhaṭīya* (c. 499 AD) refers to the value[1] as 'approximate' (*āsanna*). Nīlakaṇṭha

[1]The value given is $\frac{62832}{20000} = 3.1416$, correct to four decimal places.

while commenting upon the verse in his *Āryabhaṭīya-bhāṣya* raises the question: "Why then has an approximate value been mentioned here instead of the actual value?", and then explains [1, p. 41]:

> Given a certain unit of measurement in terms of which the diameter (*vyāsa*) specified has no [fractional] part (*niravayava*), the same measure when employed to specify the circumference (*paridhi*) will certainly have a [fractional] part (*sāvayava*). ... Even if you go a long way (i.e., keep on reducing the measure of the unit employed), the fractional part [in specifying one of them] will only become very small. A situation in which there will be no [fractional] part is impossible, and this is what is the import [of the expression *āsanna*].

Evidently, what Nīlakaṇṭha is trying to explain here is the incommensurability of the circumference and the diameter of a circle.

Sum of an infinite geometric series

In his *Āryabhaṭīya-bhāṣya*, while deriving an interesting approximation for an arc of a circle in terms of the *jyā* (Rsine) and the *śara* (Rversine),[2] Nīlakaṇṭha presents a detailed explanation of how to sum an infinite geometric series. The specific series that arises in this context is:

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots + \left(\frac{1}{4}\right)^n + \ldots = \frac{1}{3}\,. \qquad (1)$$

At the outset, Nīlakaṇṭha poses a very important question [1, p. 106]:

> How do you know that [the sum of the series] increases only upto that [limiting value] and that it certainly increases upto that [limiting value]?

Proceeding to answer the above question, Nīlakaṇṭha first obtains the sequence of results

$$\frac{1}{3} \;=\; \frac{1}{4} + \frac{1}{(4.3)}\,,$$

$$\frac{1}{(4.3)} \;=\; \frac{1}{(4.4)} + \frac{1}{(4.4.3)}\,,$$

and so on, which leads to the general result

$$\frac{1}{3} - \left[\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots + \left(\frac{1}{4}\right)^n\right] = \left(\frac{1}{4}\right)^n \left(\frac{1}{3}\right). \qquad (2)$$

Nīlakaṇṭha then goes on to present the crucial argument: As we sum more terms, the difference between $\frac{1}{3}$ and sum of powers of $\frac{1}{4}$ (as given by RHS of the above equation), becomes extremely small, but never zero. Only when we take all the terms of the infinite series together do we obtain the equality expressed in (1).

Binomial series expansion

Yuktibhāṣā [2, pp. 188-89] presents a very interesting derivation of the binomial series for $(1+x)^{-1}$ by making iterative substitutions in an algebric identity. The method given in the text may be summarized as follows.

Consider the product $a\left(\frac{c}{b}\right)$, where some quantity a is multiplied by the multiplier c, and divided by the divisor b—all assumed to be positive. This product can be rewritten as:

$$a\,\left(\frac{c}{b}\right) = a - a\,\frac{(b-c)}{b}. \qquad (3)$$

In the expression $a\frac{(b-c)}{b}$ appearing above, if we want to replace the division by b (the divisor) by division by c (the multiplier), then we have to make a subtractive correction (called *śodhya-phala*) which amounts to the following equation.

$$a\frac{(b-c)}{b} = a\frac{(b-c)}{c} - \left(a\frac{(b-c)}{c} \times \frac{(b-c)}{b}\right). \qquad (4)$$

Now, in the second term (inside parenthesis) if we again replace the division by the divisor b by the multiplier c, we have to make a subtractive-correction once again and proceeding thus we obtain a series in which all the odd terms (leaving out the first term a) will be negative and the even ones positive:[3]

$$a\frac{c}{b} \;=\; a - a\frac{(b-c)}{c} + \ldots + (-1)^{m-1}a\left[\frac{(b-c)}{c}\right]^{m-1}$$

[2]Considering a circle of radius R, if $s = R\theta$ is the arc of a circle, subtending an angle θ (in radians) at the centre, then

$$\begin{aligned} jy\bar{a}(s) &= R\sin\theta \\ \acute{s}ara(s) &= \text{Rvers}\,\theta \;=\; R(1-\cos\theta). \end{aligned}$$

[3]It may be noted that if we set $\frac{(b-c)}{c} = x$, then $\frac{c}{b} = \frac{1}{(1+x)}$. Hence, the series (5) is none other than the well known binomial series

$$\frac{a}{1+x} = a - ax + ax^2 - \ldots + (-1)^m ax^m + \ldots\,,$$

which is convergent for $-1 < x < 1$.

$$+(-1)^m a\left[\frac{(b-c)}{c}\right]^m + \ldots. \tag{5}$$

Regarding the termination of the process, both *Yuktibhāṣā* and *Kriyākramakarī* clearly mention that logically there is no end to the process. However, they note that the process may be terminated after having obtained the desired accuracy by neglecting the subsequent *phalas* as their magnitudes become smaller and smaller. In fact, *Kriyākramakarī* explicitly mentions the condition ($(b \sim c) < c$) under which the succeeding *phalas* will become smaller and smaller [4].

Estimation of sums of integral powers of natural numbers

The word employed in the Indian mathematical literature for summation is *saṅkalita*. *Yuktibhāṣā* [2, pp. 192-96] gives a general method of estimating the sum of powers of natural numbers (*sama-ghāta-saṅkalita*). The detailed procedure given in the text—which is tantamount to providing a proof by induction—may be outlined as follows. The sum of the first n natural numbers may be written as

$$\begin{aligned} S_n^{(1)} &= n + (n-1) + ... + 1 \\ &= n + (n-1) + (n-2) .... + (n-(n-1)) \\ &= n.n - [1 + 2 + ... + (n-1)]. \end{aligned} \tag{6}$$

When n is very large, the quantity to be subtracted from n^2 is practically (*prāyeṇa*) the same as $S_n^{(1)}$, thus leading to the estimate

$$S_n^{(1)} \approx n^2 - S_n^{(1)} \qquad \text{or} \qquad S_n^{(1)} \approx \frac{n^2}{2}. \tag{7}$$

The sum of the squares of the natural numbers upto n may be written as

$$S_n^{(2)} = n^2 + (n-1)^2 + ... + 1^2. \tag{8}$$

It can be easily shown that the excess of $nS_n^{(1)}$ over $S_n^{(2)}$ is

$$nS_n^{(1)} - S_n^{(2)} = S_{n-1}^{(1)} + S_{n-2}^{(1)} + S_{n-3}^{(1)} + \ldots \tag{9}$$

Recalling (7), the above equation may be written as

$$\begin{aligned} nS_n^{(1)} - S_n^{(2)} &\approx \frac{(n-1)^2}{2} + \frac{(n-2)^2}{2} + \ldots \approx \frac{S_n^{(2)}}{2} \\ \text{or} \qquad S_n^{(2)} &\approx \frac{n^3}{3}. \end{aligned} \tag{10}$$

Proceeding along similar lines, *Yuktibhāṣā* argues that

$$nS_n^{(k-1)} - S_n^{(k)} = S_{n-1}^{(k-1)} + S_{n-2}^{(k-1)} + S_{n-3}^{(k-1)} + \ldots \ . \tag{11}$$

If the lower order *saṅkalita* $S_n^{(k-1)}$ has already been estimated to be $S_n^{(k-1)} \approx \frac{n^k}{k}$, then (11) leads to

$$\begin{aligned} nS_n^{(k-1)} - S_n^{(k)} &\approx \frac{(n-1)^k}{k} + \frac{(n-2)^k}{k} \ldots \approx \frac{S_n^{(k)}}{k} \\ \text{or} \qquad S_n^{(k)} &= \sum_{i=1}^{n} i^k \approx \frac{n^{k+1}}{k+1}. \end{aligned} \tag{12}$$

Mādhava series for π

The infinite series for $\frac{\pi}{4}$ enunciated by Mādhava in the form of a verse,[4] is the well known series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots \ . \tag{13}$$

We shall now present the derivation of the above result as outlined in *Yuktibhāṣā* [2, pp. 183–98] . For this, let us consider the quadrant OP_0P_nS of the square circumscribing the given circle (see Figure 1) of radius r. Divide the side P_0P_n into n equal parts (n very large). The resulting segments P_0P_i's ($i = 1, 2, \ldots, n$) are known as the *bhujās* and the line joining its tip and the centre OP_i's are known as *karṇas*. The points of intersection of these *karṇas* and the circle are denoted by A_i. The *bhujās* P_0P_i, the *karṇas* k_i and the east-west line OP_0 form right-angled triangles whose hypotenuses are given by

$$k_i^2 = r^2 + \left(\frac{ir}{n}\right)^2. \tag{14}$$

Considering two successive *karṇas*—ith and the previous one as shown in the figure—and the pairs of similar triangles, $OP_{i-1}C_i$ and $OA_{i-1}B_i$ and $P_{i-1}C_iP_i$ and P_0OP_i, it can be shown that

$$A_{i-1}B_i = \left(\frac{r}{n}\right)\left(\frac{r^2}{k_{i-1}k_i}\right). \tag{15}$$

[4] This verse is cited by Śaṅkara Vāriyar in his commentary *Kriyākramakarī* on *Līlāvatī*:

> *vyāse vāridhinihate rūpahṛte vyāsasāgarābhihate|*
> *triśarādiviṣamasaṅkhyābhaktamṛṇaṃ svaṃ pṛthak kramāt kuryāt||*
>
> The diameter multiplied by four and divided by unity [is found and saved]. Again the products of the diameter and four are divided by the odd numbers like three, five, etc., and the results are subtracted and added in order.

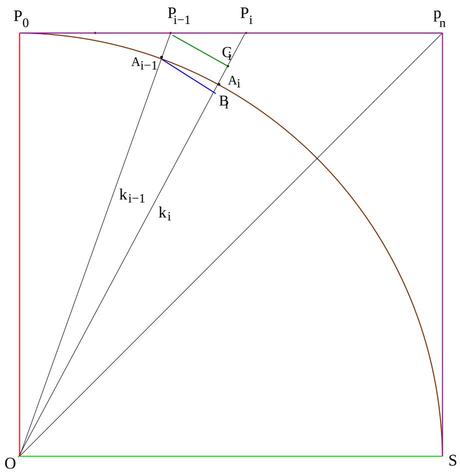

Figure 1: Geometrical construction used in the proof of the infinite series for π.

Now the text presents the crucial argument: When n is large, the Rsines $A_{i-1}B_i$ corresponding to different arc-bits $A_{i-1}A_i$ can be taken as the arc-bits themselves. Thus, $\frac{1}{8}$th of the circumference of the circle can be written as the sum of the contributions given by (15).

$$\frac{C}{8} \approx \left(\frac{r}{n}\right)\left[\left(\frac{r^2}{k_0k_1}\right) + \left(\frac{r^2}{k_1k_2}\right) + \cdots + \left(\frac{r^2}{k_{n-1}k_n}\right)\right]. \tag{16}$$

It is further argued in the text that the denominators $k_{i-1}k_i$ may be replaced by the square of either of the *karṇas* i.e., by k_{i-1}^2 or k_i^2 since the difference is negligible. Thus (16) may be re-written in the form

$$\begin{aligned}
\frac{C}{8} &= \sum_{i=1}^{n} \frac{r}{n}\left(\frac{r^2}{k_i^2}\right) \\
&= \sum_{i=1}^{n} \left(\frac{r}{n}\right)\left(\frac{r^2}{r^2 + \left(\frac{ir}{n}\right)^2}\right) \\
&= \sum_{i=1}^{n} \left[\frac{r}{n} - \frac{r}{n}\left(\frac{\left(\frac{ir}{n}\right)^2}{r^2}\right) + \frac{r}{n}\left(\frac{\left(\frac{ir}{n}\right)^2}{r^2}\right)^2 - \ldots\right]
\end{aligned} \tag{17}$$

In the series expression for the circumference given above, factoring out the powers of $\frac{r}{n}$, the summations involved are that of even powers of the natural numbers. Recalling the estimates that were obtained earlier (12) for these sums when n is large, we arrive at the result[5]

$$\frac{C}{8} = r\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right), \tag{18}$$

which is same as (13), the well known series for $\frac{\pi}{4}$.

The end-correction (*antya-saṃskāra*)

It is well known that the series for $\frac{\pi}{4}$ given by (13) is an excruciatingly slowly converging series.[6] Mādhava has found an ingenious way to circumvent this problem of slow convergence. The technique employed by Mādhava is known as *antya-saṃskāra*. The nomenclature stems from the fact that a correction (*saṃskāra*) is applied towards the end (*anta*) of the series, when it is terminated after a certain number of terms.

Suppose we terminate the series after the term $\frac{1}{p}$, and consider applying the correction-term (*antya-saṃskāra*) $\frac{1}{a_p}$, then (13) becomes

$$\begin{aligned}
\frac{\pi}{4} &\approx 1 - \frac{1}{3} + \frac{1}{5} \ldots + (-1)^{\frac{p-1}{2}}\frac{1}{p} + (-1)^{\frac{p+1}{2}}\frac{1}{a_p} \\
&= S_p + (-1)^{\frac{(p+1)}{2}}\frac{1}{a_p}.
\end{aligned} \tag{19}$$

where S_p denotes the partial sums in the Mādhava series. Three successive approximations to the correction-divisor a_p given by Mādhava may be expressed as:

$$\begin{aligned}
a_p(1) &= 2p + 2 \\
a_p(2) &= 2p + 2 + \frac{4}{(2p+2)} \\
a_p(3) &= 2p + 2 + \frac{4}{(2p+2) + \frac{16}{(2p+2)}}
\end{aligned} \tag{20}$$

Yuktibhāṣā [2, pp. 201-07] has outlined the proof of these approximations. Following the procedure outlined there, we arrive at the following continued fraction for the correction-divisor:

$$a_p = 2p + 2 + \cfrac{2^2}{(2p+2) + \cfrac{4^2}{(2p+2) + \cfrac{6^2}{(2p+2) + \ldots}}} \tag{21}$$

[5] In modern terminology, the above derivation amounts to the evaluation of the following integral

$$\frac{C}{8} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{r}{n}\right)\left(\frac{r^2}{r^2 + \left(\frac{ir}{n}\right)^2}\right) = r\int_0^1 \frac{dx}{1+x^2}.$$

[6] It is so slow that even for obtaining the value of π correct to 2 decimal places one has to find the sum of hundreds of terms and for getting it correct to 4-5 decimal places we need to consider millions of terms.

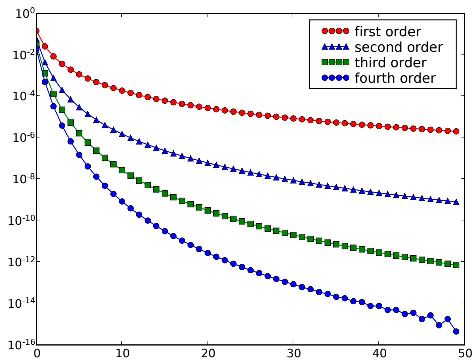

Figure 2: Error in the value of π computed using the Mādhava series with different *antya-saṃskāras*. Here x-axis represents the number of terms used in finding S_p.

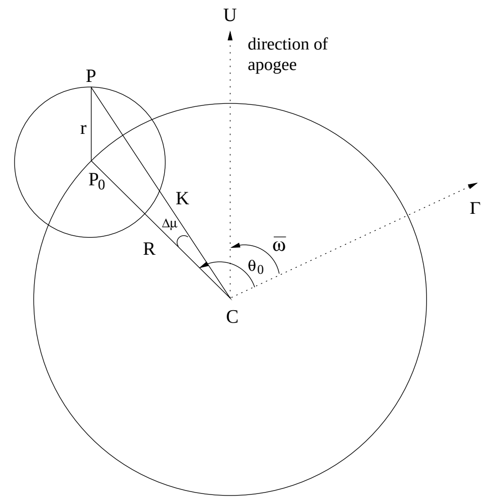

Figure 3: *Manda* correction.

The four curves depicted in Figure 2 represent the variation of the error in the computed value of π by using four successive correction-terms—the correction-divisors of three of which are given in (20).

Instantaneous velocity of a planet

The Indian astronomers, for a variety of reasons—such as determining of the exact moment of beginning and ending of a *tithis*,[7] the time of occurrence of eclipses, the conjunction of planets, and the like—were compelled to evolve better and better techniques for finding the instantaneous velocity of a planet accurately. In this connection, they have not only found the derivatives of sine and cosine function, but also of the arcsine function and the ratio of two functions that are continuously varying with time.

In Figure 3 we have depicted the motion of a planet using an epicyclic model. Here P_0 and P represent the mean and the actual planet respectively. θ_0 is the mean longitude of a planet, ϖ the longitude of the *ucca* (aphelion or apogee as the case may be) and R the radius of the concentric circle and K is the *karṇa* (hypotenuse) or the (variable) distance of the planet from the centre of the concentric C.

If r_0 and r be the mean and actual radii of the epicycle, then the *manda*-correction $\Delta\mu$—known as *manda-phala*, which is essentially the same as the equation of centre correction in modern astronomy—that needs to be applied to the mean longitude of the planet can be shown to be,[8]

$$R\sin(\Delta\mu) \quad = \quad \left(\frac{r_0}{R}\right) R\sin(\theta_0 - \varpi). \qquad (22)$$

Having noted that $\frac{r_0}{R} \ll 1$ in the case of many planets, Bhāskara II used the approximation $R\sin(\Delta\mu) \approx \Delta\mu$, and obtained the following correction which when added to the mean velocity gives the true instantaneous velocity (*tātkālika-sphuṭagati*) of the planet:

$$\frac{d}{dt}(\Delta\mu) = \left(\frac{r_0}{R}\right) R\cos(\theta_0 - \varpi)\frac{d}{dt}(\theta_0 - \varpi). \qquad (23)$$

Actually the instantaneous velocity of the planet has to be evaluated from the more accurate relation

$$\Delta\mu = R\sin^{-1}\left[\left(\frac{r_0}{R}\right) R\sin(\theta_0 - \varpi)\right]. \qquad (24)$$

The correct expression for the instantaneous velocity which involves the derivative of arc-sine function has been given by Nīlakaṇṭha[9] in his *Tantrasaṅgraha* [9]:

$$\frac{d}{dt}\left[\sin^{-1}\left(\frac{r_0}{R}\sin(\theta_0 - \varpi)\right)\right] = \frac{r_0\cos(\theta_0 - \varpi)\,\dfrac{d(\theta_0 - \varpi)}{dt}}{\sqrt{R^2 - r_0^2\sin^2(\theta_0 - \varpi)}}. \qquad (25)$$

It may be also be mentioned here that Acyuta Piśāraṭi (c. 1550–1620) a disciple of Jyeṣṭhadeva, has given

[7] It refers to the time taken by the Moon to lead the Sun exactly by 12°.

[8] Equation (22) is arrived at by imposing the constraint that the radius of the epicycle r in the *manda* process varies with the *karṇa* K such that the relation $\frac{r}{K} = \frac{r_0}{R}$ is always satisfied.

[9] In his *Jyotirmīmāṃsā*, Nīlakaṇṭha mentions that this result is due to his teacher Dāmodara.

the correct form for the derivative of the ratio of two functions—both varying with time—in his work *Sphuṭa-nirṇayatantra* [7] while calculating the instantaneous velocity of a planet in a slightly different planetary model due to Muñjāla.

Concluding remarks

It is difficult to exaggerate the key role played by calculus in the advancement of science, particularly during the 17th and 18th century. Two names generally associated with the advent of calculus are that of Newton and Leibniz, both belonging to the later part of the 17th century. Though they have played a decisive role in its growth, if one were to trace the 'long' evolution of ideas that gave birth to calculus from a historical perspective, it turns out that they neither initiated the ideas, nor its development got terminated with them.

It was amply demonstrated above that Saṅgamagrāma Mādhava—a brilliant astronomer and mathematician of the 14th century, belonging to Kerala—not only discovered the "Gregory-Leibniz" series, but also devised very efficient methods (*antya-saṃskāra*) to obtain fast convergent approximations of the same. In fact, employing a certain correction-term given by Mādhava it is possible to obtain π value correct to 13 decimal places, just by evaluating 50 terms of the "Gregory" series which converges excruciatingly slowly (see fn.6).

The sophisticated analysis, behind the technique devised by Mādhava in accelerating the convergence of a slowly converging series was not discussed in the paper as it does not fall under the scope of it. Neither did we present the brilliant derivation of the Rsine and Rcosine series due to Mādhava. Nevertheless, it would suffice to mention that mathematicians of Kerala around 14th century had clearly mastered the technique of handling the infinitesmal and the infinite—the two pillars on which the edifice of infinitesmal calculus rests upon—with great felicity. Introducing these mathematical ideas and techniques as a part of the main stream educational system, would besides being exiting and making the education more complete, may also be of great pedagogical value.

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